TOWARDS AN INTEGRAL-THEORETIC DEFINITION OF MOTIVIC BPS INVARIANTS

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Abstract. This document presents notes for a talk given during the Winter School *Motives in Montpellier* in January 2026. This conference was organised by Clément Dupont, Ulysse Mounoud, Nikola Tomić, Sofian Tur-Dorvault who I would like to thank. My talk was about a work in progress trying to relate motivic BPS invariants as introduced by Meinhardt and Reineke [MR19] to (equivariant) motivic integrals such as in the work of Loeser and Wyss [LW21]

1. Moduli stacks and BPS Sheaves

Let $k = \mathbb{C}$. Let $\mathcal{X}$ be an Artin stack which is a moduli of objects in a finite length k-linear abelian category, and such that $\mathcal{X}$ admits a good moduli space $\mathcal{X} \longrightarrow X$. In this situation, there is a perverse sheaf ϕ_{BPS} on X, called BPS sheaf, which is well-adapted to the singularities on X.

Remark 1.1. The name BPS comes from the names of physicists Bogomol'nyi, Prasad and Sommerfield, after whom so-called BPS states are named. The BPS sheaf is refining a numerical enumerative invariant, which was originally defined for moduli spaces of 1-dimensional sheaves on (Calabi-Yau) 3-folds and which is related to the count of BPS states, a string-theoretic concept.

Example 1.2. An example is given by moduli spaces of Higgs bundles on curves. Let C be a smooth projective curve. A Higgs bundle on C is a pair (E, ϕ) such that E is a vector bundle over C and $\phi \colon E \longrightarrow E \otimes K_C$. The rank r and degree d of (E, ϕ) are those of E. Fixing a slope $\mu = \frac{d}{r}$ and introducing a stability condition gives a finite length k-linear abelian category. Let $\mathcal{X}_\mu$ be the stack of semi-stable Higgs bundles of slope μ. It decomposes in connected components indexed by the rank :

$$\mathcal{X}_\mu = \bigsqcup_{r>0} \mathcal{X}_{r,d}.$$

where $d = \mu r$. Each component has a good moduli space $\pi_r \colon \mathcal{X}_{r,d} \longrightarrow X_{r,d}$.

Using ideas from Davison, Kinjo and Koseki constructed a BPS sheaf ϕ_{BPS} on the $\mathcal{X}_{r,d}$ and showed the following theorem.

Theorem 1.3 (χ-independence, [KK24])**.** *Let $d, d' \in \mathbb{Z}$ and $r > 0$. Then,*

$$H^*(X_{r,d}, \phi_{\mathrm{BPS}}) \simeq H^*(X_{r,d'}, \phi_{\mathrm{BPS}})$$

This theorem illustrates the fact that ϕ_{BPS} is well-adapted to singularities of moduli spaces, because when d, r are coprime, $X_{r,d}$ is smooth, while it can be highly singular otherwise.

The construction of ϕ_{BPS} uses involved techniques based on vanishing cycle sheaves. There is now a construction working in the much more general setting of symmetric stacks [BDIKP25].

More generally, χ-independence is an important expected property of numerical BPS invariants, which motivates the study of BPS invariants and their refinements.

2. BPS sheaves and p-adic integration

From now on we assume that $\mathcal{X}$ is a smooth Artin stack, and we spread out the moduli situation over a p-adic ring of integers $\mathcal{O}_F$ of a p-adic field F. Let $\mathbb{F}_q$ be the residue field.

We consider the $\mathbb{G}_m$-rigidification $\mathcal{X} \to \mathcal{X}^{\mathrm{rig}} \to X$, which is birational to the good moduli space x. Using the Haar measure on local F-analytic charts of X and glueing via top-forms provided by the local triviality of the relative canonical bundle $\Omega_{\mathcal{X}^{\mathrm{rig}}/X}$, Carocci-Orecchia-Wyss obtained a canonical p-adic measure μ_{can} on an (analytified) subset $X(\mathcal{O}_F)^{\natural} \subseteq X(\mathcal{O}_F)$ [COW24].

This p-adic measure is related to the BPS sheaf through the following[1].

- Consider
$$\begin{aligned} f_{\mathrm{BPS}} \colon X(\mathbb{F}_q) &\longrightarrow \mathbb{C} \\ x &\longmapsto \frac{\mathrm{Tr}(\mathrm{Frob} \mid \phi_{\mathrm{BPS},x})}{q^{\dim_x X}} \end{aligned}$$
- Consider
$$\begin{aligned} f_{\alpha} \colon X(\mathbb{F}_q) &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_{X(\mathcal{O}_F)^{\natural}_x} g_{\alpha}\mu_{\mathrm{can}} \end{aligned}$$
where $X(\mathcal{O}_F)^{\natural}_x$ is the fibre of x of the residue map and where g_{α} is a function associated to the $\mathbb{G}_m$-gerbe $\alpha \colon \mathcal{X} \to \mathcal{X}^{\mathrm{rig}}$ given by rigidification. The construction uses the Hasse invariant of the Brauer group $Br(F) \simeq \mathbb{Q}/\mathbb{Z}$.

Theorem 2.1 (Groechenig-Wyss-Ziegler [GWZ24])**.** *If $\mathcal{X}$ is a smooth Artin stack associated to a k-linear category as before, with $k = \mathbb{F}_q$ and some technical assumptions, then*
$$f_{\mathrm{BPS}} = f_{\alpha}.$$

Using this result, the authors reproved a χ-independence result for moduli spaces of 1-dimensional sheaves on del Pezzo surfaces, originally proven by Maulik-Shen [MS23]. The same strategy does not work (at the moment) for Higgs bundles, since $\mathcal{X}_{r,d}$ is not smooth.

3. Motivic BPS invariants

My goal is to update the previous relation between p-adic integrals and BPS invariants to a motivic one. For that, I follow the line of the proof of Theorem 2.1.

Step 1 : Motivic BPS **invariants.** In fact, Theorem 2.1 is based on a motivic definition of BPS invariants introduced by Meinhardt-Reineke[2] [MR19]. The motivic BPS invariant is a class
$$[\mathrm{BPS}] \in \underline{K}_0(\mathrm{Stacks}/\mathcal{X})[\mathbb{L}^{-\frac{1}{2}}]$$
where the Grothendieck ring is defined suitably given that $\mathcal{X}$ is not finite type and stacky. This modified ring $\underline{K}_0(\mathrm{Stacks}/\mathcal{X})[\mathbb{L}^{-\frac{1}{2}}]$ carries a λ-ring structure thanks to the symmetric monoidal structure of $\mathcal{X}$. The product of the λ-ring is a convolution product defined by
$$[U \to \mathcal{X}] \cdot [V \to \mathcal{X}] = [U \times V \to \mathcal{X} \times \mathcal{X} \xrightarrow{\oplus} \mathcal{X}].$$
There are also σ-operations σ_i and Adams operations ψ_i, such that $\psi_i(\mathbb{L}) = \mathbb{L}^i$. Using them, one defines mutually inverse operations called plethystic exponentials and logarithms. For a class $\mathcal{F} \in \underline{K}_0(\mathrm{Stacks}/\mathcal{X})[\mathbb{L}^{-\frac{1}{2}}]$, $\mathcal{F}(x)$ denotes the pullback $x^*\mathcal{F} \in \underline{K}_0(\mathrm{Stacks}/\kappa(x))[\mathbb{L}^{-\frac{1}{2}}]$.

[1] I took inspiration from the exposition of Paul Ziegler in his talk of 15/8/25 at the conference *Representations, Moduli and Duality* in Les Diablerets.

[2] Note that the notation for the motivic BPS invariant in this article is DT.

Definition 3.1. The class $\mathcal{I} \in \underline{K}_0(\mathrm{Stacks}/\mathcal{X})[\mathbb{L}^{-\frac{1}{2}}]$ defined on each pullback along $x \in |\mathcal{X}|$ by

$$\mathcal{I}(x) = \frac{\mathbb{L}^{\frac{-\dim_x \mathcal{X}}{2}}}{[\mathrm{Aut}(x)]}$$

is called the shifted identity motive.

Definition 3.2 ([MR19])**.** The motivic BPS invariant is the class defined by the relation

$$\frac{[\mathrm{BPS}]}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} = \mathrm{Log}(\mathcal{I})$$

where the plethystic logarithm is

$$\mathrm{Log}(\mathcal{I}) = \sum_n \frac{\mu_n}{n} \psi_n(\log(\mathcal{I})).$$

Step 2 : Inertia stacks. Now that we have a motivic definition for the BPS invariants, we would like to relate it to p-adic or motivic integrals. The intermediate step used in [GWZ24] consists in showing that $\mathrm{Log}(\mathcal{I})$ can be written in terms of volumes of inertia stacks twisted by a fermionic shift and by the rigidification gerbe α. In [GWZ24], the volume is expressed in terms of number of $\mathbb{F}_q$-points. I present a motivic analogue.

For $r \geq 1$, we define the r-th cyclotomic inertia stack as the mapping stack

$$\mathcal{Y}_r = \mathrm{Map}(B\mu_r, \mathcal{X}^{\mathrm{rig}}).$$

Concretely, a k-point in $|\mathcal{Y}_r|$ is given by a pair $y = (x, \phi)$ where $x \in \mathcal{X}^{\mathrm{rig}}(k)$ and $\phi\colon \mu_r \to \mathrm{Aut}^{\mathrm{rig}}(x)$. The automorphisms $\mathrm{Aut}(y)$ in $\mathcal{Y}_r$ are given by $\mathrm{Cent}(\phi) \subset \mathrm{Aut}^{\mathrm{rig}}(x)$.

Moreover, the gerbe α induces a local system $\mathcal{L}_\alpha$ on $\mathcal{Y}_r$, whose restriction to (x, ϕ) is determined by a character χ_α of the relative Weyl group W of $\mathrm{Cent}(\phi)$. I use Molien series to define a motivic version of the twist by the gerbe α.

Definition 3.3 (Motivic Molien series)**.** Let T be a maximal torus in $\mathrm{Cent}(\phi)$ and $\mathfrak{t}$ its Lie algebra. For $g \in W$, let $\tilde{g}$ denote its image in $GL(\mathfrak{t})$. We define the motivic Molien series by

$$[BT]_{\chi_\alpha} := \frac{1}{|W|} \sum_{g \in W} \frac{\overline{\chi_\alpha}(g)}{\det(\mathbb{L}\mathbf{1} - \tilde{g})} \in K_0^{\mathbb{Q}}(\mathrm{Stacks}/k).$$

Example 3.4. Let $T \subseteq \mathrm{Cent}(\phi) \subseteq \mathrm{Aut}^{\mathrm{rig}}(x)$ be given by $\mathbb{G}_m \subseteq \mathbb{G}_m \rtimes S_2 \subseteq \mathrm{PGL}_2$. Then, $W = S_2$, and $\chi_\alpha = \mathrm{sgn}$. We have

$$[B\mathbb{G}_m]_{\chi_\alpha} = \frac{1}{2}\Big(\frac{1}{(\mathbb{L}-1)} - \frac{1}{(\mathbb{L}+1)}\Big).$$

Note that

$$\frac{[B\mathbb{G}_m]_{\chi_\alpha}}{(\mathbb{L}-1)} = \frac{1}{2}\Big(\frac{1}{(\mathbb{L}-1)^2} - \frac{1}{(\mathbb{L}^2-1)}\Big) = \frac{1}{2}\Big([B\mathbb{G}_m^2] - \psi_2([B\mathbb{G}_m])\Big).$$

Using Molien series, I could prove a motivic analogue of [GWZ24, Theorem 5.12].

Theorem 3.5 (M., in progress)**.**

(1) The character χ_α of W has an explicit elementary description.

(2) There is an equality in $K_0^{\mathbb{Q}}(\mathrm{Stacks}/k)$:

$$\mathrm{Log}(\mathcal{I})(x) = -\lim_{T \to \infty} \sum_{r \geq 1} \sum_{y \in \mathcal{Y}_{r,x}} \frac{[BT]_{\chi_\alpha}}{(\mathbb{L}-1)} \mathbb{L}^{-\omega(y) - s(y)} T^r \tag{3.1}$$

where

- *The series in T is rational of degree* 0.
- $\omega(y)$ *is a fermionic shift, it is defined with respect to the symmetric bilinear pairing coming from* Ext *groups.*
- $s(y)$ *is an additional shift related to* $\mathrm{Cent}(\phi)$.

The right hand side of (3.1) is closely related to [GWZ24, Theorem 5.12]. Explicitely, if one substitute q for $\mathbb{L}$, one should find the same expression, which was then further related to p-adic integrals in [GWZ24], when $\mathcal{X}$ is smooth.

Step 3 : motivic integrals. A missing step is to relate the RHS of (3.1) to a motivic integral on X, as it has been done for the p-adic setting in [GWZ24]. The difficulty is to express the twist by the gerbe as a suitable motivic function, using an equivariant framework such as in [LW21].

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