Mirror Symmetry of Moduli Spaces of Higgs Bundles

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Mirror Symmetry

Mirror Symmetry Conjectures. Given a smooth compact Calabi-Yau manifold X of dimension n, there should be a mirror Calabi-Yau of same dimension $\hat{X}$ such that :

(Homological Mirror Symmetry) $Fuk(X) \simeq D^b Coh(\hat{X})$

(Topological Mirror Symmetry test) $h^{p,q}(X) = h^{n-p,q}(\hat{X})$

How to find the mirror ?

Strominger-Yau-Zaslow conjecture. A Calabi-Yau X should admit a **fibration structure** by *special Lagrangian tori*. Then $\hat{X}$ is constructed by dualizing the fibers, in the sense of **abelian varieties**.

Moduli spaces of $\mathsf{SL}_n/\mathsf{PGL}_n$ Higgs bundles provided one of the first known examples to verify (a modified version of) SYZ conjecture.

Do they also satisfy a (modified) topological mirror symmetry test ?

Let $M = [X/\Gamma]$ be an orbifold, with Γ a finite abelian group.
Let $x \in X^\gamma$. The eigenvalues of γ on $T_x X$ are denoted $\{\xi^{c_1}, \ldots, \xi^{c_n}\}$, where $0 < c_i \leq |\Gamma|$. Then the **fermionic shift** at x is $F(\gamma) = \sum_i \frac{c_i}{n}$.
The **stringy E-polynomial** is

$$E_{st}(M) = \sum_{\gamma \in \Gamma} E(X^\gamma)^\Gamma \, (uv)^{F(\gamma)}.$$

Moduli of Higgs bundles

Let $\mathcal{C}$ be a smooth projective curve of genus g over a base scheme S. Let G be a reductive group. Let $D \in \mathrm{Pic}(\mathcal{C}/S)$.

A **$G-$Higgs bundle** is a pair (P, ϕ) where

P is a principal $G-$bundle over $\mathcal{C}$.

ϕ is a global section of $\mathsf{Ad}(P) \otimes \mathcal{O}_{\mathcal{C}}(D)$.

Hitchin fibration

$$\begin{aligned} h_{GL_n} : \mathcal{M}^d_{GL_n} &\longrightarrow \mathcal{A} = \textstyle\bigoplus_{i=1}^n H^0(C, \mathcal{O}_{\mathcal{C}}(iD)) \\ (E,\phi) &\longmapsto \chi_{char}(\phi) \end{aligned}$$

$a \in \mathcal{A} \longleftrightarrow$ degree n map $\pi \ : \ \mathcal{C}' \to \mathcal{C}$ called **spectral curve**.

If $\mathcal{C}'$ is smooth, BNR correspondence states :

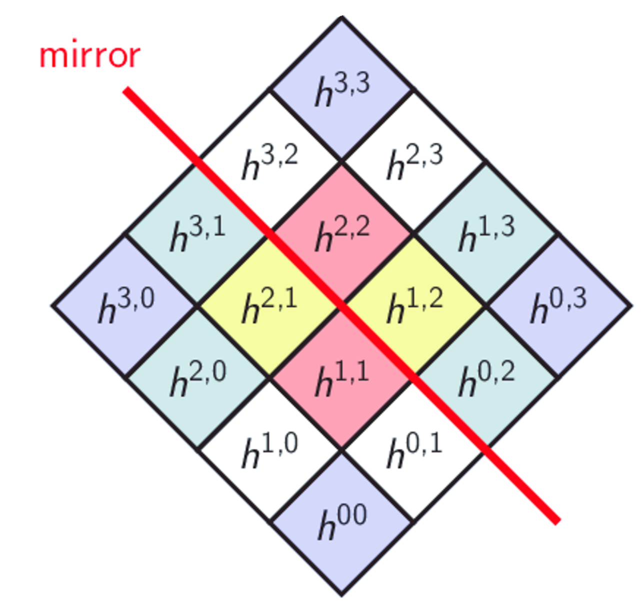
$$h^{-1}_{GL_n}(a) \simeq Pic^{d'}(\mathcal{C}') \qquad d' = d - \deg D \frac{n(n-1)}{2}.$$

For SL_n and PGL_n, smooth Hitchin fibres are torsors under dual abelian varieties called **Prym varieties**.

$$h^{-1}_{\mathsf{SL}_n}(a) \simeq \mathsf{Prym}^{d'}_{\mathsf{SL}_n} = \{L' \in \mathrm{Pic}^{d'}(\mathcal{C}') \mid \det(\pi_* L') = L\}$$

$$h^{-1}_{\mathsf{PGL}_n}(a) \simeq \mathsf{Prym}^{e'}_{\mathsf{PGL}_n} = \mathrm{Pic}^{e'}(\mathcal{C}')/\pi^*\mathrm{Pic}^0(\mathcal{C})$$

A Non-Archimedean Topological Mirror Symmetry



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$M^L_{\mathsf{SL}_n}$: semi-stable Higgs bundles (E, ϕ) of determinant $L \in \mathcal{P}ic^d(\mathcal{C})$ with $\mathsf{tr}\ \phi = 0$.

$M^e_{\mathsf{PGL}_n}$: semi-stable PGL_n-Higgs bundles of degree e.

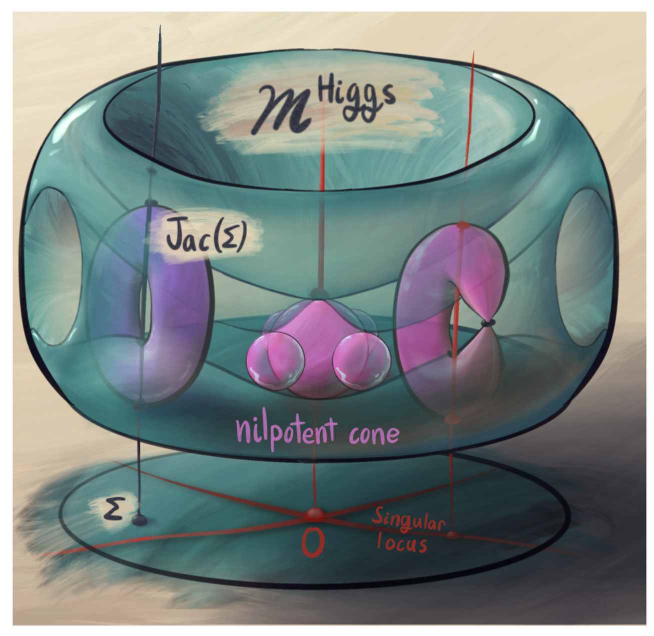
Theorem (generalising Groechenig-Wyss-Ziegler [1])

Let the base scheme be $S = Spec\ \mathcal{O}_F$ a $p-$adic ring, $D = K_{\mathcal{C}}$ or $D > K_{\mathcal{C}}$. Then,

$$\int_{M^L_{\mathsf{SL}_n}(\mathcal{O}_F)^\sharp} f^{e'}_\alpha \ \mu_{can} = \int_{M^e_{\mathsf{PGL}_n}(\mathcal{O}_F)^\sharp} f^{d'}_{\alpha_N} \ \mu_{can}.$$

The proof uses that $(M^L_{\mathsf{SL}_n}, \alpha^{e'})$ and $(M^e_{\mathsf{PGL}_n}, \alpha^{d'}_N)$ verify a form of SYZ conjecture. The torsor structure on one side is controlled by the **gerbe** on the other side.

$$\alpha \colon \mathcal{M}^L_{\mathsf{SL}_n} \to \mathcal{M}^{L,rig}_{\mathsf{SL}_n} \qquad \alpha_N \colon \mathcal{M}^e_{\mathsf{SL}_n}/\Gamma \to \mathcal{M}^{e,rig}_{\mathsf{SL}_n}/\Gamma$$



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The coprime case

Let $(d, n) = (e, n) = 1$.

$M^L_{\mathsf{SL}_n}$ is smooth, $f_\alpha \equiv 1$.

$M^e_{\mathsf{PGL}_n} = M^e_{\mathsf{SL}_n}/\Gamma$ has finite quotient singularities.

∗ Get an equality of **(twisted) finite fields counts** for the residue field k_F of F.

$$\sharp\mathcal{M}^d_{\mathsf{SL}_n}(k_F) = \sharp_{st,\alpha^{d'}_N}\mathcal{M}^e_{\mathsf{PGL}_n}(k_F)$$

Equivalent to :

$$\mathsf{tr}(\mathsf{Frob} \mid H^*(M^d_{\mathsf{SL}_n,k_F}, \mathbb{Q}_\ell)) = \mathsf{tr}(\mathsf{Frob} \mid H^*_{st}(M^e_{\mathsf{PGL}_n,k_F}, \mathcal{L}_{\alpha^{d'}_N}))$$

∗ Using it for infinitely many k_F, get an equality of **stringy $E-$polynomials**, conjectured by Hausel and Thaddeus [2].

$$\mathsf{E}(\mathcal{M}^d_{\mathsf{SL}_n}) = \mathsf{E}_{st,\alpha^{d'}_N}(\mathcal{M}^e_{\mathsf{PGL}_n})$$

Interpretations in the non-coprime case

∗ For $\mathsf{deg}\ D > 2g-2$,

$M^L_{\mathsf{SL}_n}$ is singular, $\mathcal{M}^{L,rig}_{\mathsf{SL}_n}$ smooth,

Using results by Maulik and Shen [3] we get :

Let n be an odd prime. For $\mathcal{C}$, D, L defined over k_F, the residue field with q elements of a $p-$adic ring $\mathcal{O}_F$,

$$\int_{M^L_{\mathsf{SL}_n}(\mathcal{O}_F)^\sharp} f_\alpha \ \mu_{can} = q^{-\dim M^L_{\mathsf{SL}_n}} \, \mathsf{tr}(\mathsf{Frob} \mid IH_c(M^L_{\mathsf{SL}_n,k_F}, \mathbb{Q}_\ell)).$$

∗ For $D = K_{\mathcal{C}}$, we conjecture an interpretation in terms of **BPS cohomology**.

References

[1] Michael Groechenig, Dimitri Wyss, and Paul Ziegler. “Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration”. In: *Inventiones Mathematicae* 221 (2020).
[2] Tamás Hausel and Michael Thaddeus. “Mirror symmetry, Langlands duality, and the Hitchin system”. In: *Inventiones Mathematicae* 153.1 (2003).
[3] Davesh Maulik and Junliang Shen. “On the intersection cohomology of the moduli of SL_n-Higgs bundles on a curve”. In: *Journal of Topology* 15.3 (2022).